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# VIBRATION AND STABILITY OF A CRACKED SHAFT SIMULTANEOUSLY SUBJECTED TO A FOLLOWER FORCE WITH AN AXIAL FORCE

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**Abstract**—An analysis is presented for the vibration and stability of a non-uniform shaft with a crack by use of the transfer matrix approach. The shaft is simultaneously subjected to a tangential follower force, which is distributed over the center line, with an axial force. For this purpose, the governing equations of the shaft are written as a coupled set of first order differential equations by using the transfer matrix of the shaft. Once the matrix has been determined by the numerical integration of equations, the eigenvalues of vibration and the critical flutter loads are obtained. The method is applied to shafts with linearly varying radii, subjected to a concentrated follower force, and the natural frequencies and flutter loads are calculated numerically, to provide information about the effects of them of varying cross-section, span and stiffnesses of intermediate supports, and the position and depth of the crack. © 1998 Elsevier Science Ltd. All rights reserved.

## 1. INTRODUCTION

Light weight structure have been extensively used in many industrial fields such as in mechanical, aerospace and rocket engineering, and therefore vibration and stability problems of shafts have become of increasing importance. Especially, the majority of the structural parts of machines are operated in the range of limited fatigue strength, which occurs the cracks in overstressed zone. Cracks disturb the smooth operation of the machines. They increase the vibration level and often fracture the machine unless they are detected early enough. The analysis of the dynamic characteristics of cracked structural elements is an important problem in technology. Such an analysis forms the essential technique of the diagnostics systems to detect cracks.

There is a considerable number of papers available on non-conservative instability of beams subjected to follower forces. Bolotin (1963) has extensively studied the non-conservative problems of elastic stability, detailed explanations for which are provided in the book. Neamat-Nasser (1967) and Kounadis and Katsikadelis (1976) studied the flutter loads of Beck's columns with the shear deformation and rotatory inertia taken into account. Kounadis (1977), and Kounadis and Katsikadelis (1979) have studied the stability of Timoshenko beams with attached masses subjected to a follower force, and Saito and Otomi (1979) have studied the vibration and stability of beams with an attached mass under axial and tangential loads. Irie *et al.* (1980) calculated the critical flutter loads of a Timoshenko beam of a cross-section prescribed by an arbitrary function subjected to a follower force of various types. Many researchers (see e.g. Smith and Herrmann, 1972; Sundararajan, 1974; Venkateswara and Kanaka, 1982; Lee, 1992; Lee and Yang, 1994) have analyzed the non-conservative instability of beams resting on an elastic foundation. De Rosa and Franciosi (1990), and Takahashi and Yoshioka (1996) have studied the influence of an intermediate support on the stability behavior of cantilever beams and double beams subjected to follower forces. On the other side, there are some papers on the stability problems of cracked beams, for making the diagnostic system to detect the crack. Stability of columns with single crack subjected to follower and vertical loads was studied by Anifantis and Dimarogonas (1983). Chen and Chen (1988) analyzed the vibration and stability of cracked rotating blades.

However, most of these studies have been confined to cracked beams subjected to a follower force and no papers have been presented non-conservative instability problems for non-uniform cracked shafts simultaneously subjected to a follower force with an axial force.

This paper presents an analysis of the vibration and stability of a non-uniform Timoshenko shaft with an open crack simultaneously subjected to a follower force with an axial force, in which the transfer matrix approach is used. For this purpose, the equations of motion of the shaft are written as a coupled set of first order differential equations. By introducing the transfer matrix of the shaft, these equations are expressed as a matrix differential equation. Once the transfer matrix has been determined by the numerical integration of the matrix equation, the frequency equations are expressed in terms of the elements of the matrix for a given combination of boundary conditions.

By the application of the present method, the natural frequencies (eigenvalues) and the critical flutter loads of some cantilever shafts of varying cross-section simultaneously subjected to a tangential follower force with an axial force have been calculated numerically, and discussed in what follows.

## 2. CRACK MODELING

Timoshenko shaft of radius $R$ with a transverse part-through surface crack is considered. The shaft has local flexibility due to the crack for the general loading, which was presented by Tada *et al.* (1973). For a shaft with a surface crack and loaded with bending moment and shear force, the additional displacement $u_i$ due to a crack of depth $\alpha$, the $i$ direction is given by (see e.g. Tada *et al.*, 1973; Papadopoulos and Dimarogonas, 1987)

$$u_i = \frac{\partial}{\partial P_i}\int_0^{\alpha} J(\alpha)\,\mathrm{d}\alpha \tag{1}$$

where $P_i$ is the load in the same direction as the displacement and $J$ the strain energy density function. The function is

$$J = \frac{1}{E'}\left[\left(\sum_{i=1}^{6} K_{\mathrm{I}i}\right)^2 + \left(\sum_{i=1}^{6} K_{\mathrm{II}i}\right)^2 + m\left(\sum_{i=1}^{6} K_{\mathrm{III}i}\right)^2\right] \tag{2}$$

where $E' = E$ or $E/(1-\nu^2)$ for the plane stress and plane strain, respectively, $E$ is Young's modulus. $m = 1+\nu$, $\nu$ is Poisson's ratio and $K_{ij}$ are the stress intensity factor for the $i$ = I, II, III modes and for $j = 1, 2, \ldots, 6$ the load index. In this case, the stress intensity factors for bending and shear loads, $K_{ij}$ ($ij$ = I5 and $ij$ = II3), respectively, are only used.

The local flexibility in the presence of the width $2b$ of a crack is defined by

$$C_{ij} = \frac{\partial u_{ij}}{\partial P_j} = \frac{\partial^2}{\partial P_i\,\partial P_j}\left[\int_{-b}^{b}\int_0^{\alpha} J(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}z\right] \tag{3}$$

The stress intensity factors for bending and shear loads are given by

$$\begin{aligned} K_{\mathrm{I}5} &= \sigma_5\sqrt{\pi\alpha}F_2(\alpha/h), \quad \sigma_5 = (4P_5/\pi R^2)(R^2-z^2)^{1/2} \\ K_{\mathrm{II}3} &= \sigma_3\sqrt{\pi\alpha}F_{\mathrm{II}}(\alpha/h), \quad \sigma_3 = \kappa P_3/(\pi R^2) \end{aligned} \tag{4}$$

where

$$F_2(\alpha/h) = (\tan\theta/\theta)^{1/2}[0.923+0.199(1-\sin\theta)^4]/\cos\theta$$

$$F_{\mathrm{II}}(\alpha/h) = [1.122-0.561(\alpha/h)+0.085(\alpha/h)^2+0.18(\alpha/h)^3]/(1-\alpha/h)^{1/2}$$

$$\theta = \pi\alpha/(2h), \quad h = 2(R^2-z^2)^{1/2} \tag{5}$$

Here $\kappa = 6(1+v)/(7+6v)$ is a shearing coefficient for circular cross-section (Cowper, 1966). Combining relation (2)–(4) yields the dimensionless compliance

$$\overline{c_{55}} = \pi ER^3 c_{55}/(1-v^2) = 64\int_0^{\bar{\alpha}}\int_0^{\bar{b}} \bar{y}(1-\bar{z}^2)F_2^2(\bar{h})\,\mathrm{d}\bar{z}\,\mathrm{d}\bar{y}$$

$$\overline{c_{33}} = \pi ERc_{33}/(1-v^2) = 4\int_0^{\bar{\alpha}}\int_0^{\bar{b}} \bar{y}F_{\mathrm{II}}^2(\bar{h})\,\mathrm{d}\bar{z}\,\mathrm{d}\bar{y}$$

$$\bar{z} = z/R, \quad \bar{y} = y/R, \quad \bar{h} = y/h, \quad \bar{b} = b/R, \quad \bar{\alpha} = \alpha/R \tag{6}$$

## 3. FUNDAMENTAL EQUATIONS

We consider a non-uniform Timoshenko shaft of length $l$ with two intermediate supports. The origin $o$ is taken at one end of the shaft, and the shear center axis is taken as the $x$-axis. With the rotary inertia and shear deformation taken into account, the equations of flexural motion of the shaft when subjected to a tangential follower force $f(x)$, which is distributed over the axis, with a compressive axial force $f_1(x)$ can be written as (see e.g. Irie *et al.*, 1980)

$$\frac{\partial Q^*}{\partial x} - f(x)\frac{\partial^2 w^*}{\partial x^2} + \rho A(x)\omega^2 w^* = 0 \tag{7}$$

$$Q^* - \frac{\partial M^*}{\partial x} + f_1(x)\frac{\partial w^*}{\partial x} + \rho I(x)\omega^2\psi^* = 0 \tag{8}$$

where $\rho$ is the mass per unit volume, $A(x)$ is the cross-sectional area, and $I(x)$ is the second moment of area of the shaft. The variables $w^*$ and $\psi^*$ denote the transverse deflection and the slope due to pure bending, respectively. The variable $\omega$ is the natural frequency. The bending moment $M^*$ and shear force $Q^*$, respectively, are given by

$$M^* = -EI(x)\frac{\partial\psi^*}{\partial x} \tag{9}$$

$$Q^* = \{\kappa GA(x)+f(x)\}\left(\frac{\partial w^*}{\partial x} - \psi^*\right) - f_1(x)\psi^* \tag{10}$$

where $E$ is Young's modulus and $G$ is the shear modulus.

For simplicity of the analysis, the following dimensionless variables are introduced:

$$w^* = wl, \quad \psi^* = \psi, \quad Q^* = \frac{EI_0}{l^2}Q, \quad M^* = \frac{EI_0}{l}M$$

$$\xi = \frac{x}{l}, \quad \bar{a} = \frac{A(x)}{A_0}, \quad \bar{i} = \frac{I(x)}{I_0}$$

$$s_0^2 = \frac{A_0 l^2}{I_0} = \left(2\frac{l}{R_0}\right)^2, \quad p = \frac{l^2}{EI_0}f(x) \quad p_1 = \frac{l^2}{EI_0}f_1(x) \tag{11}$$

Here $A_0$ and $I_0$ are the sectional area and the second moment of area at one end ($x = 0$). The value $s_0$ is the slenderness ratio at one end. The quantities without an asterisk (*) are the respective dimensionless variables. As a frequency parameter

$$\lambda^4 = \frac{\rho A_0 l^4 \omega^2}{E I_0} \tag{12}$$

is used here.

Equations (7)–(10) for the shaft are written as a matrix differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\{Z(\xi)\} = [U(\xi)]\{Z(\xi)\} \tag{13}$$

where the state vector $\{Z(\xi)\} = \{w\psi QM\}^{\mathrm{T}}$ and the coefficient matrix $[U(\xi)]$ is

$$U_{11} = 0, \quad U_{12} = 1+\frac{p_1}{g}, \quad U_{13} = \frac{1}{g}, \quad U_{14} = 0$$

$$U_{21} = 0, \quad U_{22} = 0, \quad U_{23} = 0, \quad U_{24} = -\frac{1}{\bar{i}}$$

$$U_{31} = \frac{-\lambda^4 \bar{a}}{\left(1-\dfrac{p}{g}\right)}, \quad U_{32} = \frac{-p}{\left(1-\dfrac{p}{g}\right)}\frac{p_1 g'}{g^2}, \quad U_{33} = \frac{-p}{\left(1-\dfrac{p}{g}\right)}\frac{g'}{g^2}$$

$$U_{34} = \frac{-p}{\left(1-\dfrac{p}{g}\right)}\left(\frac{1}{\bar{i}}+\frac{p_1}{g\bar{i}}\right)$$

$$U_{41} = 0, \quad U_{42} = \bar{i}\frac{\lambda^4}{s_0^2}+p_1+\frac{p_1^2}{g}, \quad U_{43} = 1+\frac{p_1}{g}, \quad U_{44} = 0$$

$$\left(\because g = \kappa\frac{G}{E}s_0^2\bar{a}+p, \quad \frac{G}{E} = \frac{1}{2(1+\nu)}\right) \tag{14}$$

in which the prime denotes differentiation with respect to $\xi$.

## 4. ANALYSIS BY USE OF THE TRANSFER MATRIX METHOD

Since analytical solutions of eqn (13) cannot be obtained for a shaft with a surface crack of varying cross-section, the transfer matrix approach is adopted here. In general, the state vector $\{Z(\xi)\}$ can be expressed as

$$\{Z(\xi)\} = [T(\xi)]\{Z(0)\} \tag{15}$$

by using the transfer matrix $[T(\xi)]$ of the shaft. From eqns (13) and (15), the following equation is derived:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}[T(\xi)] = [U(\xi)][T(\xi)] \tag{16}$$

For a shaft of varying cross-section, the matrix $[T(\xi)]$ is obtained by integrating eqn (16) numerically with the starting value $[T(0)] = [1]$ (the unit matrix), which is given by taking $\xi = 0$ in eqn (15). In the calculation, the elements of the transfer matrix are determined numerically by using Runge–Kutta integration method.

The continuity at intermediate supports (at $\xi = \beta_i$, $i = 1, 2$) yields

$$\{Z_R(\beta_i+0)\} = [P]_{(\beta_i)}\{Z_L(\beta_i-0)\} \tag{17}$$

where

$$[P]_{(\beta_i)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ k_i & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

The continuity conditions at the position (at $\xi = \xi_c$) of the crack which is assumed to be opened are

$$\{Z_R(\beta_i+0)\} = [P_c]_{(\xi_c)}\{Z_L(\beta_i-0)\} \tag{19}$$

where

$$[P_c]_{(\xi_c)} = \begin{bmatrix} 1 & 0 & e_1 r^3 \bar{c}_{33} & 0 \\ 0 & 1 & 0 & -e_2 r \bar{c}_{55} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e_1 = \frac{(1-\nu^2)}{4}\left(\frac{R_0}{l}\right)^3 \quad e_2 = \frac{(1-\nu^2)}{4}\left(\frac{R_0}{l}\right) \quad r = \frac{R}{R_0} \tag{20}$$

At an arbitrary position, the state vector of the shaft is expressed as

$$\{Z(\xi)\} = [M]_{(\xi)}\{Z(0)\} \tag{21}$$

where the final transfer matrix of the shaft ($\beta_1 > \xi_c > \beta_2 > \xi$) is

$$[M]_{(\xi)} = [T(\xi)][P]_{(\beta_2)}[T(\beta_2)][P_c]_{(\xi_c)}[T(\xi_c)][P]_{(\beta_1)}[T(\beta_1)] \tag{22}$$

This method can be applied to any combination of boundary conditions of the shaft. Here, a free-elastically restrained shaft will be discussed. For a cantilever shaft elastically restrained against transverse and rotational motions by springs of the dimensionless stiffnesses $k$ and $k'$, respectively, at one end

($\xi = 0$), the boundary conditions are

$$Q-kw = 0, \quad M+k'\psi = 0 \quad \text{at } \xi = 0$$
$$Q = 0, \quad M = 0 \quad \text{at } \xi = 1$$
$$\left(k = \frac{Kl^3}{EI_0}\right), \quad \left(k' = \frac{K'l}{EI_0}\right) \tag{23}$$

The substitution of eqns (21) into (23) yields the frequency equation

$$\begin{bmatrix} \frac{1}{k}M_{31}+M_{33} & -\frac{1}{k'}M_{32}+M_{34} \\ \frac{1}{k}M_{41}+M_{43} & -\frac{1}{k'}M_{42}+M_{44} \end{bmatrix}_{(1)} \begin{Bmatrix} Q \\ M \end{Bmatrix}_{(0)} = 0 \tag{24}$$

with only the elements of $[M]_{(1)}$ necessary for the calculation.

Since $[U(\xi)]$ and $[T(\xi)]$ depend on the frequency parameter $\lambda$, $[M]_{(1)}$ is also a function of $\lambda$. The natural frequencies of the shaft are determined by calculating the eigenvalues $\lambda$ of eqn (24).

## 5. NUMERICAL CALCULATION AND DISCUSSION

In this section, the method is applied to some cracked shafts of varying cross-section, and the eigenvalues of vibration and the critical flutter loads are calculated numerically. Consider a shaft whose radius is expressed as

$$R(x) = R_0-(R_0-R_1)\left(\frac{x}{l}\right) \tag{25}$$

where $R_0$ and $R_1$ denote the radii of one and other end, respectively. In this case, $\bar{a}$ and $\bar{i}$ are written as

$$\bar{a} = \left\{1-\left(1-\frac{R_1}{R_0}\right)\xi\right\}^2$$

$$\bar{i} = \left\{1-\left(1-\frac{R_1}{R_0}\right)\xi\right\}^4 \tag{26}$$

Tangential forces of the following types are considered.

A concentrated follower force-Beck's problem (1952): when a concentrated follower force $F_B$ acts at the free end, $p$ is written as

$$p = p_B \quad (p_B = F_B l^2/EI_0) \tag{27}$$

where $p_B$ denotes the dimensionless force parameter.

The numerical calculations were carried out for the cracked cantilever shafts subjected to a concentrated follower force. The origin of the axis is taken at clamped end. The eigenvalues of vibration and the critical flutter loads were numerically obtained for the above mentioned shafts and the results are displayed in what follows.

Figure 1 shows the eigenvalue curves of cantilever shafts subjected to a concentrated follower force with an axial force at the free end. The compressive axial force expresses as the plus value in this case. The values of the curves on the ordinate indicate the eigenvalues of the shaft without the action of the follower force. With increasing force, the eigenvalues of the first mode increase, while those of the second mode decrease. The maxima of the branch of the eigenvalue curves indicate critical flutter loads $p_{B,f}$ beyond which the natural frequencies become complex quantities and therefore the motion becomes an unstable vibration with exponentially increasing amplitude.

The critical flutter loads and eigenvalues of the first and second modes increase with increasing tensile axial force under the buckling load. That is the tensile axial force stabilizes the cantilever shaft.

The critical flutter loads of tapered shafts subjected to a concentrated force are shown in Fig. 2. The critical load for the crack depth $\alpha/2R = 0$ indicates for the shaft without a

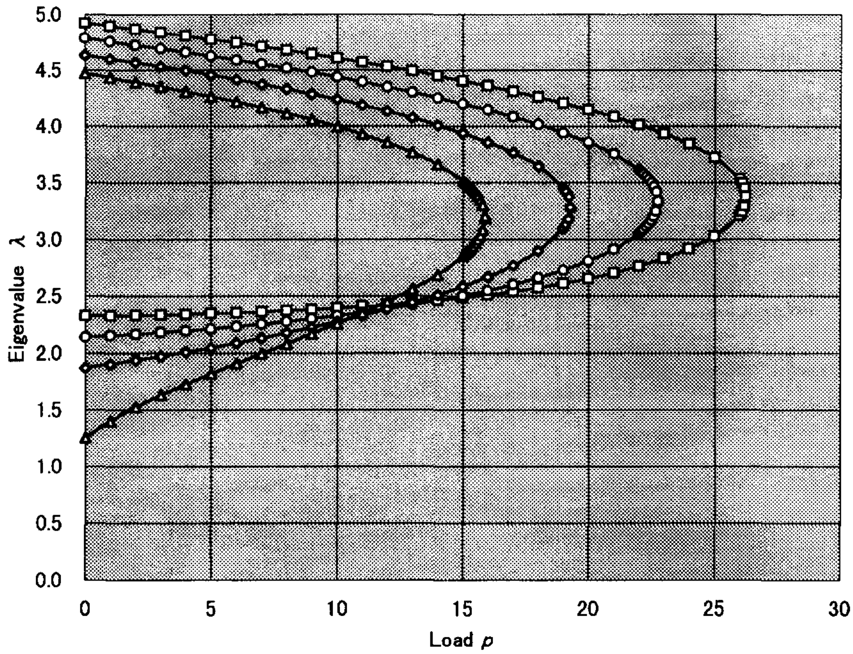


Fig. 1. Eigenvalue curves of cantilever shafts subjected to a concentrated follower force with an axial force. ($\nu = 0.3$, $s_0 = 50$, $R_1/R_0 = 1.0$, $\alpha/2R = 0.0$, $k_1 = k_2 = 0.0$, $p_1$: □, $-4$; ○, $-2$; ◇, 0; △, 2.)

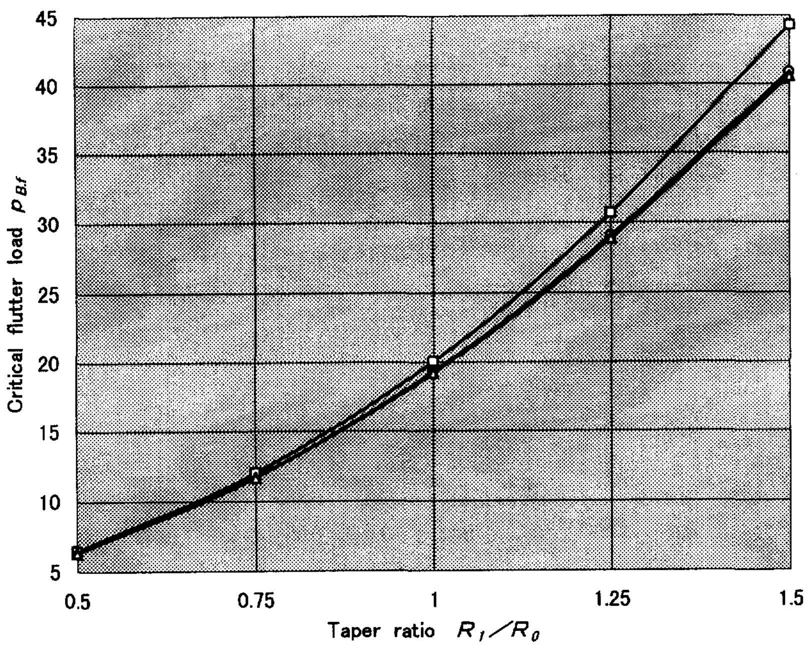


Fig. 2. Relationship between taper ratio and critical flutter loads of cracked cantilever shafts subjected to a concentrated follower force. ($\nu = 0.3$, $s_0 = 50$, $\alpha/2R = 0.3$, $k_1 = k_2 = 0.0$, $p_1 = 0$, $\xi_c$: □, 0.25; △, 0.75; ○, $\alpha/2R = 0$.)

crack and its value is almost the same as that for the crack position $\xi_c = 0.75$. The critical flutter load becomes monotonically larger with increasing taper ratio $R_1/R_0$.

The relationship between the axial force $p_1$ and the critical flutter load of cantilever shafts under a concentrated force is shown in Fig. 3. The critical flutter loads for the crack

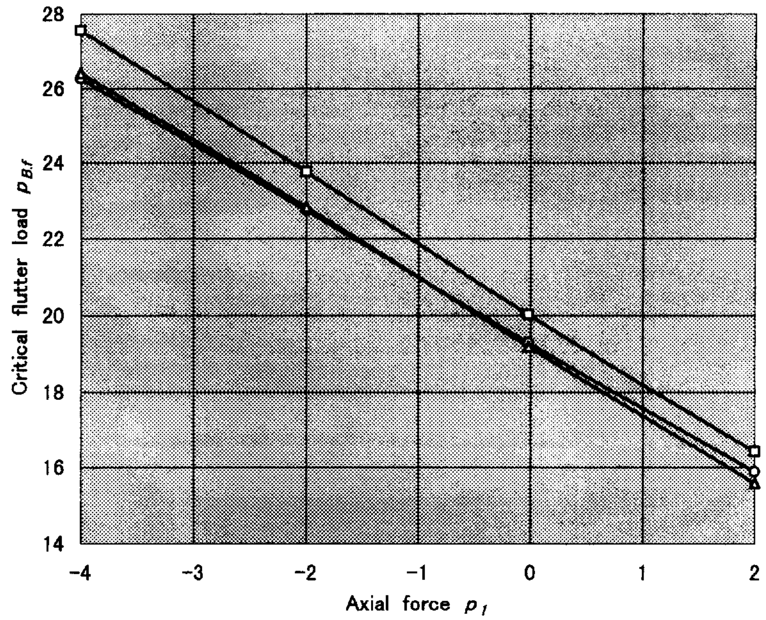


Fig. 3. Relationship between axial force and critical flutter loads of cracked cantilever shafts subjected to a concentrated follower force with an axial force. ($\nu = 0.3$, $s_0 = 50$, $\alpha/2R = 0.3$, $R_1/R_0 = 1.0$, $k_1 = k_2 = 0.0$, $\xi_c$: □, 0.25; △, 0.75; ○, $\alpha/2R = 0$.)

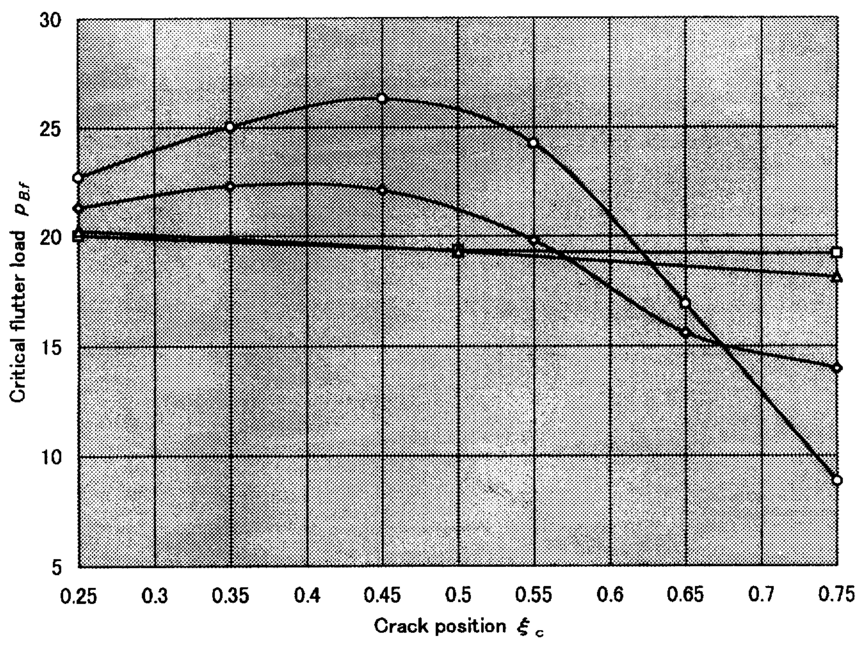


Fig. 4. Relationship between stiffnesses of supports and critical flutter loads of cracked cantilever shafts with intermediate supports subjected to a concentrated follower force. ($\nu = 0.3$, $s_0 = 50$, $R_1/R_0 = 1.0$, $\alpha/2R = 0.3$, $\beta_1 = \xi_c - 0.1$, $\beta_2 = \xi_c + 0.1$, $p_1 = 0$, $k_1 = k_2$: □, 0; △, 10; ◇, 50; ○, 100.)

positions $\xi_c = 0.25$ and 0.75 linearly decrease with increasing compressive axial load. The critical flutter load for the crack depth $\alpha/2R = 0$ nearly equals that for $\xi_c = 0.75$.

Figure 4 shows the relationship between crack position and critical flutter loads of the shafts with intermediate supports at ($\beta_1 = \xi_c - 0.1$, $\beta_2 = \xi_c + 0.1$). The critical flutter loads for stiffnesses $k_1 = k_2 = 0$ and 10 slightly decrease as the crack position shifts to the free end. On the other hand, the critical loads for stiffnesses $k_1 = k_2 = 50$ and 100 increase up to around $\xi_c = 0.45$ and then rapidly decrease as the crack position shifts to the free end.

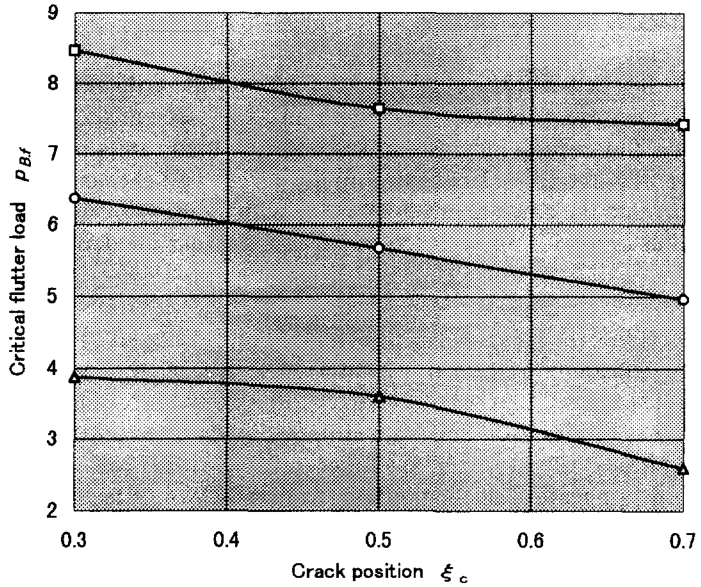


Fig. 5. Relationship between crack position and critical flutter loads of cracked cantilever shafts with intermediate supports subjected to a concentrated follower force with an axial force. ($\nu = 0.3$, $s_0 = 50$, $R_1/R_0 = 0.5$, $\alpha/2R = 0.3$, $k_1 = k_2 = 10.0$, $\beta_1 = \xi_c - 0.1$, $\beta_2 = \xi_c + 0.1$, $p_1$: □, −1; ○, 0; △, 1.)

This means that the optimal crack position exists for the various stiffnesses of intermediate supports.

Figure 5 shows the relationship between crack position and critical flutter loads of the shafts subjected to a concentrated follower force with an axial force at the free end. The shaft has two intermediate supports at ($\beta_1 = \xi_c - 0.1$, $\beta_2 = \xi_c + 0.1$). As shown in Fig. 5, the critical flutter load increases in the order of increasing tensile axial force. The critical load

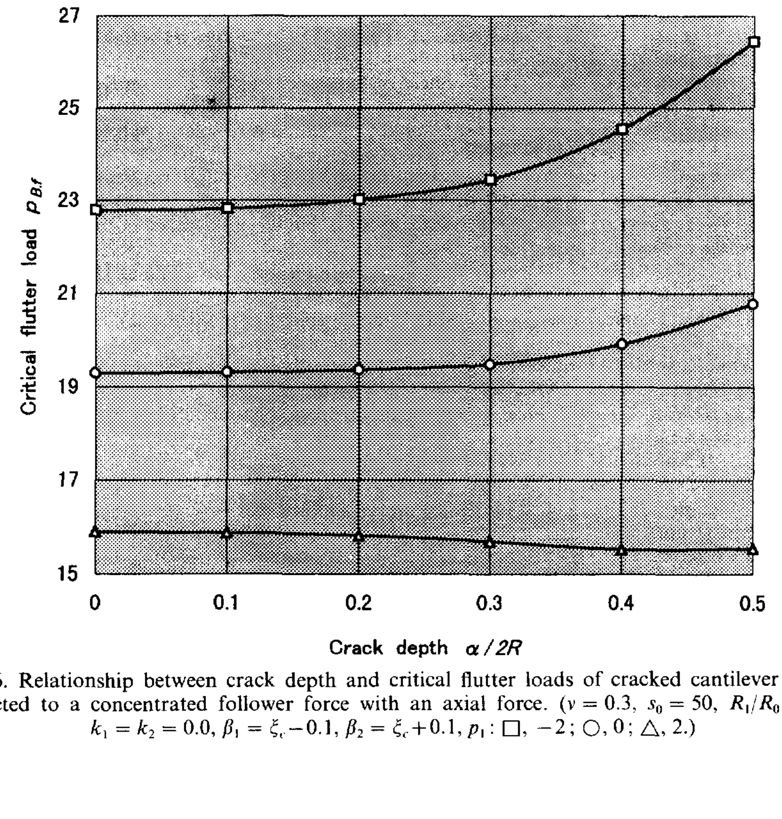


Fig. 6. Relationship between crack depth and critical flutter loads of cracked cantilever shafts subjected to a concentrated follower force with an axial force. ($\nu = 0.3$, $s_0 = 50$, $R_1/R_0 = 1.0$, $k_1 = k_2 = 0.0$, $\beta_1 = \xi_c - 0.1$, $\beta_2 = \xi_c + 0.1$, $p_1$: □, −2; ○, 0; △, 2.)

has the same tendency as shown in Fig. 4: its value slightly decreases as the crack position shifts to the free end.

The relationship between the crack depth $\alpha/2R$ and the critical flutter loads of shafts subjected to a concentrated follower force with an axial force at the free end is shown in Fig. 6. The critical flutter load increases in the order of axial forces $p_1 = 2, 0$ and $-2$. The critical loads for $p_1 = 0$ and $-2$ become rapidly larger and the load for $p_1 = 2$ slightly decreases from around the crack depth $\alpha/2R = 0.2$ with increasing crack depth. This means that the critical flutter load is not affected by the crack under the depth 0.2.

## 6. CONCLUSIONS

A transfer matrix analysis of the non-conservative instability of a cracked non-uniform Timoshenko shaft subjected to a follower force with an axial force has been developed.

The eigenvalues of vibration and critical flutter loads of a cantilever shaft of varying cross-section simultaneously subjected to a follower force with an axial force were calculated numerically by the application of the method, from which the effects of the cross-section, crack depth, crack position and the stiffness of the support have been quantitatively illustrated.

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